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POLYNOMIALS OF BEST APPROXIMATION IN $C[-1,1]$

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ABSTRACT

Equivalences between the condition $|P_n^{(k)}(x)| \leqq K(n^{-1}\sqrt{1-x^2}+1/n^2)^k n^{-\alpha}$, where $P_n(x)$ is the best n-th degree polynomial approximation to $f(x)$, and the Peetre interpolation space between $C[-1,1]$ and the space $(1-x^2)^k f^{(2k)}(x) \in C[-1,1]$ is established. A similar result is shown for $E_n(f) = \inf_{P_n} \|f-P_n\|_{C[-1,1]}$. Rates other than $n^{-\alpha}$ are also discussed.

1. Introduction

Derivatives of polynomials of best approximation were recently investigated by M. Hasson [2] and D. Leviatan [4]. The result obtained [4, Theorem 4] for P_n satisfying $\|f-P_n\| = E_n(f) \equiv \inf_{Q\in\mathcal{P}_n}\|f-Q\|$ in the $C[-1,1]$ norm is

$$(1.1)\qquad |P_n^{(k)}(x)| \leqq K|\Delta_n(x)|^{-k}\omega_r(f,1/n) \qquad \text{for } k \geqq r,$$

where $\Delta_n(x) \equiv n^{-1}\sqrt{1-x^2}+n^{-2}$ and $\omega_r(f,h)$ is the r modulus of continuity, that is $\omega_r(f,h) \equiv \sup_{0\leqq t\leqq h}\{|\Delta_t^r f(x)|;\ [x-\frac{1}{2}rh, x+\frac{1}{2}rh] \subset [-1,1]\}$,

$$\Delta_h f(x) \equiv f(x+h/2) - f(x-h/2) \quad \text{and} \quad \Delta_h^r f(x) \equiv \Delta_h(\Delta_h^{r-1} f(x)).$$

While it is clear from the term $\Delta_n(x)^{-k}$ that some care was taken to treat the behaviour near ± 1, no such scrutiny is evident in the term $\omega_r(f,1/n)$. It is known, and will be clear from this paper as well, that the polynomial of best approximation is not as sensitive to smoothness near ± 1 as it is to smoothness inside the interval $[-1+\delta, 1-\delta]$, for instance.

We will obtain the estimate

$$(1.2)\qquad |P_n^{(k)}(x)| \leqq K|\Delta_n(x)|^{-k}\omega_r^*(f,1/n)$$

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where

$$(1.3)\qquad \omega_r^*(f,h)=\sup_{0<t<h}\{|\Delta_{t\gamma(x)}^r f(x)|;[x-rt\gamma(x)/2,x+rt\gamma(x)/2]\subset[-1,1]\}$$

and $\gamma(x)\equiv\sqrt{1-x^2}$. If we examine the functions $f(x)=(1-x^2)^{\alpha/2}$, $\alpha<1$, for instance, we will find that D. Leviatan's estimate yields $|P_n^{(k)}(x)|\leqq K(\Delta_n(x))^{-k}n^{-\alpha/2}$ and our estimate $|P_n^{(k)}(x)|\leqq K(\Delta_n(x))^{-k}n^{-\alpha}$ (both for $k\geqq 1$). The expression $\omega_r^*(f,h)$ was studied in detail as the K-functional of an interpolation space, together with other related K-functionals, in [1, section 3]. In contrast, one has $\|f-P_n(x)\|\leqq K\omega_r^*(f,1/n)$. This latter type of result for L_p, $1\leqq p<\infty$ (which is the more difficult case), was announced by V. Totik in [8]. V. Totik has an additional nontrivial term in the definition of ω_r^* which is the result of treating the L_p case (see also [1]); for L_∞ no such term is needed. Moreover, $\|f-P_n(x)\|_C\leqq K\omega_r^*(f,1/n)$ follows from a result of K. Ivanov [3] relating to the moduli of continuity $\tau_k(f,\Delta_n(x))_{p',p}$ for $p'=1$, $p=\infty$ elaborated on by many Bulgarian mathematicians in dozens of papers and given by

$$(1.4)\qquad \tau_k(f,\Delta_n(x))_{p',p}=\left\|\left\{\frac{1}{2\Delta_n(x)}\int_{-\Delta_n(x)}^{\Delta_n(x)}|\Delta_v^k f(x)|^{p'}dv\right\}^{1/p'}\right\|_p$$

where the L_p norm should be taken in $[\alpha_n,\beta_n]$, where α_n and β_n are the solutions of $x\pm\frac{1}{2}r\Delta_n(x)=\pm 1$ inside $[-1,1]$. The results presented here are in a sense best possible. That is, if $|P_n^{(r)}(x)|\leqq K(\Delta_n(x))^{-r}\phi(n)$ where $\phi(n)$ is decreasing, $\phi(n)=o(1)$, and satisfies some additional conditions, then $\omega_r^*(f,1/n)\leqq M\phi(n)$. This is the analogue to the Sunouchi–Zamanski theorem, see [6] and [9], that states (for L_p) that for T_n, the trigonometric polynomial of best approximation, $\|T_n^{(r)}\|\leqq Mn^{r-\alpha}$ for $\alpha<r$ if and only if $f\in\mathrm{Lip}^*\alpha$. (The result is valid for $\alpha=r$ and for $\phi(n)\neq n^{-\alpha}$.)

In fact, this inverse theorem is an easy corollary of the asymptotic behaviour of the derivatives of the polynomials of best approximation. Hasson [2] brought an example to show that $\|f-P_n\|\sim 1/n$ and $|P_n(x)|\sim K\log n$ occur for the same $f(x)$. From the present study this is shown to be natural, as $\|f-P_n\|\leqq Kn^{-r}$ is equivalent to $\omega_{r+1}^*(f,1/n)\leqq Mn^{-r}$ and $|P_n^{(r)}(x)|\leqq K(\Delta_n(x))^{-r}n^{-r}$ is equivalent to $\omega_r^*(f,1/n)\leqq Mn^{-r}$ and even in the interval $[-1+\delta,1-\delta]$ one should expect a logarithmic term (using a Marchaud-type inequality).

2. Derivatives of P_n, the polynomial of best approximation

For $E_n(f)$ given by $E_n(f)\equiv\inf_{P\in\mathcal{P}_n}\|f-P\|_{C[-1,1]}$ we have:

THEOREM 2.1. *If for some integer r and decreasing sequence* $\phi(n)$,

$$\sum_{k=0}^{l} 2^{kr}\phi(2^k) \leqq M2^{lr}\phi(2^l) \quad \text{and} \quad E_n(f) \leqq \phi(n),$$

then for P_n, *the polynomial satisfying* $\|f - P_n\| = E_n(f)$,

$$|P_n^{(r)}(x)| \leqq M_1(\Delta_n(x))^{-r}\phi(n). \tag{2.1}$$

In particular, if for some r,

$$\sum_{k=0}^{l} 2^{kr}E_{2^k}(f) \leqq M2^{lr}E_{2^l}(f),$$

then

$$|P_n^{(r)}(x)| \leqq M_1(\Delta_n(x))^{-r}E_n(f). \tag{2.2}$$

REMARK. The use of $\phi(n)$ other than $E_n(f)$ will prove useful later. The second statement of the theorem is just a special case which for the time being is the most important. It should be noted that if $E_n(f) = n^{-r}$, the result above does not imply (1.2) and Theorem 2.1 will be used as a step in proving (1.2).

PROOF. For $2^k \leqq n < 2^{k+1}$ we write

$$P_n(x) - P_1(x) = \sum_{l=0}^{k} (P_{2^{l+1}}(x) - P_{2^l}(x)) + (P_n(x) - P_{2^{k+1}}(x)).$$

We have $P_n^{(r)}(x) = P_n^{(r)}(x) - P_1^{(r)}(x)$ for $r > 1$, while for $r = 1$, $P_1'(x)$ is bounded by $C\|f\|$ (C fixed and independent of f), and since $P_n'(x) - P_1'(x)$ would tend to infinity under the condition of our theorem, $P_1'(x)$ may be neglected. We can now write

$$\|P_{2^{l+1}}(x) - P_{2^l}(x)\| \leqq E_{2^{l+1}}(f) + E_{2^l}(f) \leqq 2\phi(2^l) \text{ and } \|P_n(x) - P_{2^{k+1}}(x)\| \leqq 2\phi(n).$$

We will now need the following Lemma, which is probably known but, as I cannot find a reference (for $r > 1$), I will include a short proof after proving Theorem 2.1.

LEMMA 2.2. *For* Q_m, *a polynomial of degree m,*

$$|Q_m^{(r)}(x)| \leqq M_r\left(\frac{m}{\sqrt{1-x^2}}\right)^r \|Q_m\| \equiv M_r\left(\frac{m}{\gamma(x)}\right)^r \|Q_m\| \tag{2.3}$$

where M_r *is independent of* Q_m, m *and* x.

Using Lemma 2.2,

$$|P_n^{(r)}(x)| \leqq M_r\left(\frac{1}{\gamma(x)}\right)^r\left\{\sum_{l=0}^{k+1}(2^{l+1})^r 2\phi(2^l) + n^r 2\phi(n)\right\}$$

$$\leqq M_r\left(\frac{1}{\gamma(x)}\right)^r \{2M2^{r(k+2)}\phi(2^{k+1}) + 2n^r\phi(n)\}$$

$$\leqq M_r K\left(\frac{1}{\gamma(x)}\right)^r n^r\phi(n).$$

Using the technique of Theorem 6 in [5, p. 41] on $P_n^{(r)}$, which is a polynomial of degree $n - r$, we have (for $n > 2r$ but otherwise the result is trivial)

$$|P_n^{(r)}(x)| \leqq K_1 n^r \phi(n)(\sqrt{1-x^2} + 1/n)^{-r}.$$

Proof of Lemma 2.2. For $r = 1$, (2.3) is well-known [5, Th. 3, p. 39]. Assume by induction that it is known for $r = k$. Using the estimate

$$|Q_m^{(k)}(x)| \leqq M_r\left(\frac{m}{\gamma(x)}\right)^k \|Q_m\| \qquad \text{for } |x_0| < 1 \quad \text{and} \quad \frac{-1-|x_0|}{2} \leqq x \leqq \frac{1+|x_0|}{2},$$

we have

$$\|Q_m^{(k)}(x)\|_{C[(-1-|x_0|)/2,(1+|x_0|)/2]} \leqq M_k\left(\frac{m}{\sqrt{1-((|x_0|+1)/2)^2}}\right)^k \|Q_m\|_{C[-1,1]}.$$

We now use the Lemma for $r = 1$ where $[-1,1]$ is replaced by the interval $(-1-|x_0|)/2 \leqq x \leqq (1+|x_0|)/2$ and obtain

$$|Q_m^{(k+1)}(x)| \leqq M_k M_1\left(\frac{m}{\sqrt{1-((|x_0|+1)/2)^2}}\right)^k\left(\frac{m}{\sqrt{((|x_0|+1)/2)^2 - x^2}}\right)\|Q_m\|.$$

For $x = x_0$ the above yields

$$|Q_m^{(k+1)}(x_0)| \leqq M_k M_1\left(\frac{2m}{\gamma(x_0)}\right)^k\left(\frac{2m}{\gamma(x_0)}\right)\|Q\| \leqq M_k M_1 2^{k+1}\left(\frac{m}{\gamma(x_0)}\right)^{k+1}\|Q_m\|.$$

This being valid for all x_0 implies our result.

Actually we proved also the following result which will be useful later.

Lemma 2.3. *Suppose for a polynomial Q_m of degree m, $|Q_m(x)| \leqq M/(1-x^2)^{l/2}$, then $|Q_m'(x)| \leqq M_1 m/(1-x^2)^{(l+1)/2}$*

As a corollary of Theorem 2.1 we can deduce the inverse theorem for polynomials of best approximation.

THEOREM 2.4. *Suppose $E_n(f)\leqq K\phi(n)$, $\sum_{k=0}^{l}2^{k(r+1)}\phi(2^k)\leqq M2^{l(r+1)}\phi(2^l)$ and $\phi(n)$ is decreasing, then $\omega^*_{r+1}(f,1/n)\leqq K_1\phi(n)$.*

PROOF. Using Theorem 2.1, we have $|P_n^{(r+1)}(x)|\leqq C(\Delta_n(x))^{-r-1}\phi(n)$. We can now write, for $x\pm\frac{1}{2}(r+1)t\gamma(x)\in[-1,1]$,

$$|\Delta^{r+1}_{t\gamma(x)}f(x)|\leqq|\Delta^{r+1}_{t\gamma(x)}(f(x)-P_n(x))|+|\Delta^{r+1}_{t\gamma(x)}P_n(x)|\leqq 2^{r+1}E_n(f)+|\Delta^{r+1}_{t\gamma(x)}P_n(x)|.$$

Using Taylor's formula with integral remainder and the estimate above of $|P_n^{(r+1)}(u)|$, we obtain, for $x\pm\frac{1}{2}(r+1)t\gamma(x)\in[-1,1]$,

$$I_n(t,x)=|\Delta^{r+1}_{t\gamma(x)}P_n(x)|\leqq C_1\sup_{|\alpha|\leqq(r+1)/2}\left|\int_{x-\alpha t\gamma(x)}^{x}(u-x+\alpha t\gamma(x))^r\,|P_n^{(r+1)}(u)|\,du\right|.$$

For $-1+(r+1)t\gamma(x)\leqq x\leqq 1-(r+1)t\gamma(x)$ and u between x and $x-\alpha t\gamma(x)$ where $|\alpha|\leqq(r+1)/2$, we have $C_2\Delta_n(x)\leqq\Delta_n(u)\leqq C_3\Delta_n(x)$ where C_2 and C_3 do not depend on x and therefore, for those x and for $t\leqq 1/n$,

$$I_n(t,x)\leqq C_4(\Delta_n(x))^{-r-1}\phi(n)(\alpha t)^{r+1}\gamma(x)^{r+1}\leqq C_5\phi(n).$$

For $x\leqq -1+(r+1)t\gamma(x)$ or $x\geqq 1-(r+1)t\gamma(x)$ and $t\leqq 1/n$ we use $(\Delta_n(x))^{-1}<n^2$ and obtain

$$I_n(t,x)\leqq C_6(n^2)^{r+1}\phi(n)(\alpha t)^{r+1}(\sqrt{1-x^2})^{r+1}\leqq C_7\phi(n),$$

which together with earilier estimates concludes the proof.

We also have the following result.

COROLLARY 2.5. *Suppose $E_n(f)\leqq K\phi(n)$, $\phi(n)$ is decreasing and $\sum_{k=0}^{l}2^{kr}\phi(2^k)\leqq M2^{lr}\phi(2^l)$, then $\omega^*_r(f,1/n)\leqq K_1\phi(n)$.*

PROOF. We may use the same proof, as we can derive from Theorem 2.1 the estimate $|P_n^{(r)}(x)|\leqq C(\Delta_n(x))^{-r}\phi(n)$. The result also follows from the relation between $\omega^*_r(f,1/n)$ and $\omega^*_{r+1}(f,1/n)$, but it seems in the present context more natural to use the estimate on $|P_n^{(r)}(x)|$.

3. The best polynomial approximation

A simple estimate for $E_n(f)$ is given in the following theorem.

THEOREM 3.1. *For $f\in C[-1,1]$ and $E_n(f)=\sup_{P\in\mathcal{P}_n}\|f-P\|$, we have*

$$E_n(f)\leqq K\omega^*_r(f,1/n). \tag{3.1}$$

REMARK. In [8], V. Totik states (without proof) for $E_n(f,p)\equiv$

$\sup_{P\in\mathcal{P}_n}\|f-P\|_{L_p}$ that

$$E_n(f,p)\leqq K\omega_\phi^r(f,1/n)_p\equiv K\left(\sup_{h\leqq n^{-1}}\|\Delta_{h\gamma(x)}^r f\|_{L_p}+\sup_{h\leqq n^{-2}}\|\Delta_h^r f\|_{L_p}\right).$$

For $p=\infty$ one observes that the second term in the definition of $\omega_\phi^r(f,1/n)_p$ is dropped. Obviously, the case $p=\infty$ is the easiest and (3.1) follows $\omega_\phi^r(f,1/n)\leqq K_1\omega_r^*(f,1/n)$. Moreover, it was shown by K. Ivanov (see [3]) in a series of papers using the moduli of continuity $\tau_r(f,t)_{p',p}$ investigated by numerous Bulgarian mathematicians that among other things $E_n(f)\leqq K\tau_r(f,\Delta_n(x))_{1,\infty}$ where $\tau_r(f,t)_{p',p}$ is given by (1.4) and it is not difficult to show that

$$\tau_r(t,\Delta_n(x))_{1,\infty}\leqq\tau_r(f,\Delta_n(x))_{\infty,\infty}\leqq K\omega_r^*(f,1/n).$$

The above is valid, although the L_∞ norm for $\tau_r(f,\Delta_n(x))_{p',\infty}$ is taken in $[\alpha_n,\beta_n]$ where α_n, β_n are the solutions of $x\pm\frac{1}{2}r\Delta_n(x)=\pm1$ and in the present discussion the L_∞ norm is taken in $[\gamma_n,\delta_n]$ where γ_n, δ_n are solutions $x\mp\frac{1}{2}r\gamma(x)=\mp1$ of, and obviously $[\gamma_n,\delta_n]$ contains $[\alpha_n,\beta_n]$. The well-known estimates on $|f(x)-P_n(x)|$ (see [7] and [5]) for some $P_n(x)$ treat the case of uniform smoothness and non-uniform convergence of $|f(x)-P_n(x)|$. (This is in contrast to uniform convergence and non-uniform smoothness here.) In that case $P_n(x)$ is not the best polynomial approximation to $f(x)$ in $C[-1,1]$. For completeness we will give a straightforward proof of the Ivanov–Totik result for $\omega_r^*(f,1/n)$. This proof will not depend on the properties of $\tau_k(f,\Delta_n(x))_{p',p}$; in fact we will not get involved with those moduli.

For the proof of Theorem 3.1 as well as for later theorems we will need the characterization of the K-functionals given in [1, Th. 3.1] with translation of the singularity to -1 and 1 where $\alpha=1/2$. For this we recall the K-functionals $K_r(t^r,f)$,

$$K_r(t^r,f)\equiv\inf_{f_1+f_2=f}\left(\|f_1\|_{C[-1,1]}+t^r\|(1-x^2)^{r/2}f_2^{(r)}(x)\|_{C[-1,1]}\right),\tag{3.2}$$

where $f_1\in C[-1,1]$ and f_2 and its first $r-1$ derivatives are locally absolutely continuous in $(-1,1]$ and $(1-x^2)^{r/2}f_2^{(r)}(x)$ is continuous in $[-1,1]$. (It would not make any difference if we just assume $(1-x^2)^{r/2}f_2^{(r)}(x)$ is in L_∞.) With the above setting [1, Th. 3.1] will yield:

THEOREM A. *Suppose $f\in C[-1,1]$, then*

$$M_1\omega_r^*(f,t)\leqq K_r(t^r,f)\leqq M_2\omega_2^*(f,t)\tag{3.3}$$

where $\omega_r^*(f,)$ and $K_r(t^r,f)$ are given in (1.3) *and* (3.2) *respectively.*

We will need for Theorem 3.2 as well as some subsequent theorems two properties of $\omega_r^*(f,h)$ given in the following lemma:

LEMMA 3.2. *For $\omega_r^*(f,h)$ given in (1.3) we have*

$$\omega_r^*(f,h) \leqq 2\omega_{r-1}^*\left(f, \frac{4}{3}\cdot\frac{r}{r-1}h\right) \qquad (\text{for } h < 1/2(r-1)) \tag{3.4}$$

and

$$\omega_r^*(f,2h) \leqq M(r)\omega_r^*(f,h) \tag{3.5}$$

where $M(r)$ is independent of f and h.

REMARK. While it is obvious that $M(r) \geqq 2^r$, $M(r)$ may be actually bigger than 2^r.

PROOF OF LEMMA 3.2. To prove (3.4) we observe that

$$\omega_r^*(f,h) \equiv \sup_{0<t<h}\left\{|\Delta_{t\gamma(x)}^r f(x)|; \left[x-\frac{r}{2}t\gamma(x), x+\frac{r}{2}t\gamma(x)\right] \subset [-1,1]\right\}$$

$$\leqq \sup_{0<t<h}\left\{\left|\Delta_{t\gamma(x)}^{r-1} f\left(x+\frac{t}{2}\gamma(x)\right)\right| + \left|\Delta_{t\gamma(x)}^{r-1} f\left(x-\frac{t}{2}\gamma(x)\right)\right|;\right.$$

$$\left.\left[x-\frac{r}{2}t\gamma(x), x+\frac{r}{2}t\gamma(x)\right] \subset [-1,1]\right\}.$$

For $\xi = x \pm t\gamma(x)/2$ and $-1 + rt\gamma(x)/2 \leqq x \leqq 1 - rt\gamma(x)/2$ we have

$$1-\xi^2 \leqq \frac{r}{r-1}(1-x^2).$$

This follows, easily for $-\frac{1}{2} < x < \frac{1}{2}$ (at least for $t < 1/2(r-1)$). For x such that $x \geqq \frac{1}{2}$ we have $\xi_\pm \equiv x \pm t\gamma(x)/2$ and trivially $1-\xi^2 \leqq 1-x^2$. We now write

$$(1-\xi_+^2) = (1-\xi_+)(1+\xi_+) \leqq \frac{r}{r-1}(1-x)(1+\xi_+) \leqq 2\frac{r}{r-1}(1-x) \leqq \frac{4}{3}\frac{r}{r-1}(1-x^2).$$

Similarly, we treat $x \leqq -1/2$, which now implies (3.4). We now use the definition of $K_r(t^r,f)$ and the fact that we have $K_r((2t)^r,f) \leqq 2^r K_r(t^r,f)$, which we combine with (3.3) to achieve the estimate

$$\omega_r^*(f,2t) \leqq \frac{1}{M_1}K_r((2t)^r,f) \leqq \frac{2^r}{M_1}K_r(t^r,f) \leqq \frac{M_2}{M_1}2^r\omega_r^*(f,t),$$

and that is (3.5) with $M(r) = (M_2/M_1)2^r$.

The following Lemma will be the crucial step in proving Theorem 3.1.

LEMMA 3.3. *Suppose for some odd r, $g,\dots,g^{(r-1)}$ are locally absolutely continuous in $(-1,1)$, $g^{(r)}(x)$ continuous in $(-1,1)$ and $\|(1-x^2)^{r/2}g^{(r)}(x)\|_{C[-1,1]}\leqq M$, then there exists a polynomial $P_n(x)$ such that $\|g-P_n\|_{C[-1,1]}\leqq MLn^{-r}$ where L depends only on r.*

REMARK. The lemma is valid for even r as well, and this will be proved later because for this we will use Theorem 3.1 and in the proof of Theorem 3.1 we need our lemma at least for odd r.

PROOF OF LEMMA 3.3. We may assume that the $g(0)=g'(0)=\cdots=g^{(r-1)}(0)=0$; otherwise we just consider $g_1(x)=g(x)-Q_{r-1}(x)$ where Q_{r-1} is a polynomial of order $r-1$, $g_1^{(r)}(x)=g^{(r)}(x)$ and $g_1^{(i)}(0)=0$ for $0\leqq i<r$. We have

$$g^{(r-1)}(x)=\int_0^x g^{(r)}(u)du \quad \text{and} \quad |g^{(r-1)}(x)|\leqq M\int_0^{|x|}\frac{du}{(1-u^2)^{r/2}},$$

which implies for $r\geqq 3$

$$(1-x^2)^{(r-2)/2}|g^{(r-1)}(x)|\leqq MR_1.$$

Similarly, for $i<r/2$,

$$(1-x^2)^{(r-2i)/2}|g^{(r-i)}(x)|\leqq MR_i,$$

and for $i>r/2$ $(r\geqq 1)$,

$$|g^{(r-i)}(x)|\leqq MR_i.$$

As r is odd, $i=r/2$ is not possible. In fact, the reason for assuming r odd is that otherwise we would get for $i=r/2$ a term with logarithmic behaviour which would interfere in our estimates. We now define $F(t)=g(\cos t)$ for $0\leqq t\leqq\pi$. The derivatives of F in $(0,\pi)$ are:

$$F'(t)=-\sin t g'(\cos t),\quad F''(t)=\sin^2 t g''(\cos t)-\cos t g'(\cos t),$$

$$F^{(3)}(t)=-\sin^3 t g^{(3)}(\cos t)+3\sin t\cos t g''(\cos t)+\sin t g'(\cos t),$$

etc. In general we have

$$F^{(2l-1)}(t)=\sum_{j=1}^{2l-1}\phi_{2l-j,l}(t)g^{(2l-j)}(\cos t)$$

and

$$F^{(2l)}(t)=\sum_{j=0}^{2l-1}\psi_{2l-j,l}(t)g^{(2l-j)}(\cos t).$$

While $\phi_{2l-1,l}(t)=(-\sin t)^{2l-1}$, $\phi_{1,l}(t)=\pm\sin t$, $\psi_{2l,l}(t)=(\sin t)^{2l}$ and $\psi_{1,l}(t)=\pm\cos t$, the exact expressions of $\phi_{j,l}$ and $\psi_{j,l}$ are quite complicated. We can, however, see that the lowest power of $\sin t$ in $\phi_{2l-1-j,l}(t)$ and $\psi_{2l-j,l}(t)$ is $(\sin t)^{2l-1-2j}$ and $(\sin t)^{2l-2j}$ for $2l-1-2j>0$ and $2l-2j\geqq 0$ respectively. Therefore,

$$|\phi_{2l-1-j,l}(t)|\leqq C(l)|\sin t|^{2l-1-2j} \quad \text{and} \quad |\psi_{j,l}(t)|\leqq C(l)|\sin t|^{2l-2j}$$

for those j. It is easy to observe that for other j, $|\psi_{j,l}(t)|\leqq C(l)$. More important is the observation that for $2l-1-2j<0$, $|\phi_{2l-1-j,l}(t)|\leqq C(l)|\sin t|$. (For other j's we showed a better estimate.) This follows the fact that $\phi_{2l-1-j,l}(t)$ is composed of elements of the type $\sin t\cdot T(t)$ for some trigonometric polynomial $T(t)$ and elements of the type

$$I(t)=\left(\frac{d}{dt}\right)^{k_r}\left\{(\sin t)^{m_r}\left(\frac{d}{dt}\right)^{k_{r-1}}\left\{(\sin t)^{m_{r-1}}\cdots\left(\frac{d}{dt}\right)^{k_1}(\sin t)^{m_1}\right\}\cdots\right\}$$

where $\Sigma k_i+\Sigma m_i$ is $2l-1$, i.e. odd. (We also know that $m_1\geqq 1$ and that $\Sigma m_i=2l-j-1$ and $\Sigma k_i=j$, but that latter information would not add anything there.) As a power series, $I(t)$ is odd or even with $\Sigma k_i+\Sigma m_i$ and therefore odd, but $I(t)$ is a combination of powers of $\sin t$ and $\cos t$ and therefore, using also the same argument at $(t-\pi)$, a multiple of $\sin t$.

We examine the function $F(t)$ and its derivatives. As

$$(1-x^2)^{(r-2i)/2}|g^{(r-i)}(x)|\leqq MR_i,$$

$F^{(r)}(t)$ is bounded in $(0,\pi)$ and so are $F^{(j)}(t)$ for $j<r$. Moreover, for an odd number $2l-1$, $2l-1<r$ we will show $F^{(2l-1)}(0+)=F^{(2l-1)}(\pi-)=0$. For terms $\phi_{2l-j-1}(t)g^{(2l-1-j)}(\cos t)$ where $2l-1-2j>0$, we have, using $|(\sin t)^{r-2j}g^{(r-j)}(\cos t)|\leqq MR_j$,

$$\begin{aligned}|(\sin t)^{2l-1-2j}g^{(2l-1-j)}(x)|&=|(\sin t)^{r-2(r-2l+1)-2j}g^{(r-(r-2l+1)-j)}(x)|\,|\sin t|^{r-2l+1}\\&\leqq MR_{r-2l+1+j}|\sin t|^{r-2l+1}.\end{aligned}$$

For terms $\phi_{2l-j-1}(t)g^{(2l-1-j)}(\cos t)$ where $2l-1-2j<0$,

$$|g^{(2l-1-j)}(x)|=|g^{(r-i)}(x)|\leqq M, \qquad \text{for } i=(r-2l+1)+j;$$

since

$$i=\frac{r}{2}+\frac{1}{2}(r-2l+1)+\left(j-\frac{2l-1}{2}\right)>\frac{r}{2}$$

we recall that the coefficient of $g^{(2l-1-j)}(\cos t)$ in that case was bounded by $C|\sin t|$ and therefore $|\phi_{2l-j-1}(t)g^{(2l-1-j)}| < K|\sin t|$. For this reason $F(t)$ can be defined on $[-\pi, \pi]$ as a periodic function by $F(-t) = F(t)$ which preserves derivatives in both edges of the interval $[0, \pi)$ (as the same argument is valid for π). We can find a trigonometric polynomial $T_n(t)$ such that

$$\|F(t) - T_n(t)\|_{C[-\pi,\pi]} \leqq C\|F^{(r)}\| n^{-r}.$$

Moreover, since $F(t) = F(-t)$, we can choose $T_n(t)$ as an even trigonometric polynomial (the De la Vallee Poussin operator

$$V_{[n/2]}(F, t) = 2\sigma_{2[n/2]}(F, t) - \sigma_{[n/2]}(F, t)$$

would be such a choice). The even trigonometric polynomial $T_n(t)$ can be written as $P_n(\cos t)$, a polynomial in $\cos t$. We complete the proof of the lemma by recalling that

$$\|F^{(r)}(t)\|_{C[0,\pi]} \leqq B(r)\|(1-x^2)^{r/2} g^{(r)}(x)\|_{C(-1,1)}.$$

Proof of Theorem 3.1. Using Lemma 3.2 it will be sufficient to prove (3.1) for odd r (and in fact for any subsequence of the integers). While it is puzzling to me, it is nevertheless easier in the proof of Lemma 3.3 to assume that r is odd. With the aid of Lemma 3.3, we now choose $f_{1,n}$ and $f_{2,n}$ such that

$$\|f_{1,n}\|_{C[-1,1]} + n^{-r}\|(1-x^2)^{r/2} f_{2,n}^{(r)}(x)\|_{C[-1,1]} \leqq 2K_r(n^{-r}, f).$$

We now choose the polynomial P_n to fit $g = f_{2,n}(x)$ in Lemma 3.3; the constant M for that g will be $2n^r K_r(n^{-r}, f)$, and therefore

$$\|f - P_n\| \leqq \|f_{1,n}\| + \|f_{2,n} - P_n\| \leqq 2K_r(n^{-r}, f) + 2n^r K_r(n^{-r}, f) L n^{-r}$$
$$\leqq (2+2L)K_r(n^{-r}, f) \leqq (2+2L)M_2\omega_r^*(f, 1/n).$$

4. Corollaries and extensions

In this section we shall be able to extend Lemma 3.3 and Theorem 2.1 just by applying Theorem 3.1, to the particular situation.

Lemma 4.1. *In Lemma 3.3 we may drop the condition that r is odd.*

Proof. If $\|(1-x^2)^m g^{(2m)}(x)\|_{C[-1,1]} \leqq M$, we can show $\omega^*_{2m+1}(g, h) \leqq Mk \cdot h^{2m}$. This follows using the expansion of g by Taylor's formula with integral remainder involving the $2m$ derivative in the expression $\Delta^{2m+1}_{t\gamma(x)} g(x)$. Using Theorem 3.1, we obtain our result.

THEOREM 4.2. *For $P_n(x)$ the best polynomial approximation in $[-1,1]$ to f, we have*

$$|P_n^{(r)}(x)| \leqq M(r)(\Delta_n(x))^{-r}\omega_r^*(f,1/n). \tag{4.1}$$

PROOF. Using Theorem 3.1, we have $E_n(f) \leqq C\omega_r^*(f,1/n)$. Using Lemma 3.2 we have $\phi(2^k) \leqq M(r)^{l-k}\phi(2^l)$ where $\phi(n) = \omega_r^*(f,1/n)$. Choosing s such that $M(r)2^{-r-s+1} \leqq 1$ we have

$$\sum_{k=0}^{l} 2^{k(r+s)}\phi(2^k) \leqq \sum_{k=0}^{l} 2^{l(r+s)}2^{(k-l)}[2^{(k-l)(r+s-1)}M^{l-k}]\phi(2^l)$$
$$\leqq 2\,2^{l(r+s)}\phi(2^l).$$

(For $\phi(n) \sim n^\alpha$ where $\alpha \leqq r$ or $\phi(n) \sim n^\alpha \log n^\beta$ where $\alpha \leqq r$, $s = 1$ is sufficient.) Therefore, using Theorem 2.1, we have

$$|P_n^{(r+s)}(x)| \leqq K(\Delta_n(x))^{-r-s}\omega_r^*(f,1/n). \tag{4.2}$$

To show that (4.1) is satisfied, we have to show that (4.2) for $s = m+1$ implies (4.2) for $s = m$. We now write

$$\left|\left(\frac{\gamma(x)}{n}\right)^{r+m} P_n^{(r+m)}(\xi)\right| = |\Delta^{r+m}_{\gamma(x)/n}P_n(x)| = |\Delta^{r+m}_{\gamma(x)/n}(P_n - f)| + |\Delta^{r+m}_{\gamma(x)/n}f(x)|$$

$$\leqq 2^{r+m}K\omega_r^*(f,1/n) + \omega_{r+m}^*(f,1/n) \leqq 2^{r+m}K\omega_r^*\left(f,\frac{1}{n}\right) + 2^m\omega_r^*\left(f,\left(\frac{4}{3}\frac{r+m}{r}\right)^m\frac{1}{n}\right)$$
$$\leqq L(r)\omega_r^*\left(f,\frac{1}{n}\right)$$

for some ξ satisfying $x-(r+m)\gamma(x)/2n < \xi < x+(r+m)\gamma(x)/2n$. We now write

$$I_n(\xi,x) = \left|\left(\frac{\gamma(x)}{n}\right)^{r+m}(P_n^{(r+m)}(\xi) - P_n^{(r+m)}(x))\right|$$
$$\leqq \left(\frac{\gamma(x)}{n}\right)^{r+m}\left|\int_x^\xi P_n^{(r+m+1)}(u)du\right|$$
$$\leqq \left(\frac{\gamma(x)}{n}\right)^{r+m}\omega_r^*(f,1/n)K\left|\int_x^\xi \Delta_n(u)^{-r-m-1}du\right|.$$

For $1-|x| \geqq (r+m)\gamma(x)/n$,

$$I_n(\xi,x) \leqq K\left(\frac{\gamma(x)}{n}\right)^{r+m}\omega_r^*(f,1/n)\frac{\gamma(x)}{2n}\left(\frac{n}{\gamma(x)}\right)^{r+m+1} \leqq K\omega_r^*(f,1/n)$$

and for $1-|x|\leqq(r+m)\gamma(x)/n$ (or $\gamma(x)\leqq Cn^{-2}$),

$$I_n(\xi,x)\leqq K\left(\frac{\gamma(x)}{n}\right)^{r+m}\omega_r^*(f,1/n)\frac{\gamma(x)}{n}(n^2)^{r+m+1}\leqq K_2\omega_r^*(f,1/n).$$

Therefore

$$\left|\left(\frac{\gamma(x)}{n}\right)^{r+m}P_n^{(r+m)}(x)\right|\leqq K_3\omega_r^*(f,1/n)$$

and using the fact that we can interpolate the polynomial $P_n^{(r+m)}$ at the zeros of the Chebychev polynomial of order $n-r-m$ as was done in [4, p. 41], we get

$$|P_n^{(r+m)}(x)|\leqq M(\Delta_n(x))^{-r-m}\omega_r^*\left(f,\frac{1}{n}\right).$$

5. The inverse result for derivatives

In this section we will show that in some sense Theorem 4.2 is best possible.

THEOREM 5.1. *Suppose $P_n(x)$ is the polynomial of best approximation of degree n, $|P_n^{(r)}(x)|\leqq M(\Delta_n(x))^{-r}\phi(n)$ with $\phi(n)=o(1)$, $n\to\infty$, $\phi(n)$ decreasing, then if $\Sigma(\phi(n)/n)<\infty$, $\omega_r^*(f,n^{-1})\leqq K\Sigma_{k=1}^{\infty}\phi(2^kn)$, and if $\Sigma_{k=1}^{\infty}\phi(2^kn)\leqq L\phi(n)$, we have $\omega_r^*(f,n^{-1})\leqq K_1\phi(n)$.*

REMARK. Only for very slowly decreasing sequences $\phi(n)$ do we have $\Sigma(\phi(n)/n)=\infty$. For instance, $\phi(n)=n^{-\alpha}$, $\alpha\leqq r$ satisfies both conditions, and $|P_n^{(r)}(x)|\leqq M(\Delta_n(x))^{-r}n^{-\alpha}$ implies $\omega_r^*(f,1/n)\leqq K_1n^{-\alpha}$.

PROOF. Using $|P_n^{(r)}(x)|\leqq M(\Delta_n(x))^{-r}\phi(n)$ we write $\|P_{2n}-P_n(P_{2n})\|\leqq M_*\phi(2n)$ where $P_n(P_{2n})$ is the best n-th degree polynomial approximation to P_{2n}. We observe that

$$M_*\phi(2n)\geqq\|P_{2n}-P_n(P_{2n})\|\geqq\|f-P_n(P_{2n})\|-\|f-P_{2n}\|\geqq E_n(f)-E_{2n}(f).$$

This implies

$$E_n(f)=\sum_{k=0}^{\infty}E_{n2^k}(f)-E_{n2^{k+1}}(f)\leqq M_*\sum_{k=1}^{\infty}\phi(2^kn).$$

The sum on the right is convergent if and only if $\Sigma_{n=1}^{\infty}(\phi(n)/n)$ is. Moreover, we can now write

$$|\Delta_{\gamma(x)/n}^r(f(x))|=|\Delta_{\gamma(x)/n}^r(f(x)-P_n(x))|+|\Delta_{\gamma(x)/n}^rP_n(x)|$$

$$\leqq 2^rE_n(f)+|P_n^{(r)}(\xi)|\left(\frac{\gamma(x)}{n}\right)^r$$

for some $x - r\gamma(x)/2n < \xi < x + r\gamma(x)/2n$. Using Lemma 2.3, we have $|P_n^{(r+1)}(x)| \leqq C(\Delta_n(x))^{-r-1}\phi(n)$ and therefore

$$|P_n^{(r)}(\xi) - P_n^{(r)}(x)| \left(\frac{\gamma(x)}{n}\right)^r \leqq \left(\frac{\gamma(x)}{n}\right)^r \left| \int_\xi^x |P^{(r+1)}(u)| du \right| \leqq C\phi(n),$$

which implies

$$|P_n^{(r)}(\xi)| \left(\frac{\gamma(x)}{n}\right)^r \leqq \left(\frac{\gamma(x)}{n}\right)^r |P_n^{(r)}(x)| + C\phi(n) \leqq C_1\phi(n).$$

The second result of the theorem is useful but is just an immediate consequence of the first.

6. Conclusions

A simple version of our theorems for $\phi(n) = n^{-\alpha}$ can be summarized by the following corollary:

COROLLARY 6.1. *Suppose* $E_n(f) = \inf_{Q \in \mathscr{P}_n} \|f - Q\|$ *and* P_n *satisfies* $E_n(f) = \|f - P_n\|$, *then*

(a) *for* $\alpha \leqq r$

$$|P_x^{(r)}(x)| \leqq C(\Delta_n(x))^{-rn^{-\alpha}} \qquad \textit{if and only if } \omega_r^*(f, 1/n) \leqq An^{-\alpha};$$

(b) *for* $\alpha < r$

$$E_n(f) \leqq Cn^{-\alpha} \qquad \textit{if and only if } \omega_r^*(f, 1/n) \leqq An^{-\alpha}.$$

REMARK. In particular, $|P_n^{(r)}(x)| \leqq C(\Delta_n(x))^{-r}n^{-r}$ is equivalent to $\omega_r^*(f, 1/n) \leqq An^{-r}$ and $E_n(f) \leqq Cn^{-r}$ is equivalent to $\omega_{r+1}^*(f, 1/n) \leqq An^{-r}$. The above provide an explanation for the result of Hasson [2, Th. 4.1], that there exists a function for which $E_n(f) \leqq \lambda/n$ and $\|P_n'(x)\|_{C[-1+\delta, 1-\delta]} \geqq K \log n$. We remember that $\omega_{r+1}^*(f, 1/n) \leqq An^{-r}$ implies $\omega_r^*(f, 1/n) \leqq A_1 n^{-r} \log n$ and that is the best estimate for some functions, even on $(-1+\delta, 1-\delta)$. For $\phi(n) = n^{-r} \log n$ we have by Theorem 4.2 and 5.1 $\omega_r^*(f, 1/n) \sim A_1 n^{-r} \log n$ is necessary and sufficient to $|P_n^{(r)}(x)| \leqq C(\Delta_n(x))^{-r}n^{-r} \log n$.

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